Supplement 5-A

Uncertainty Relations

In our discussion of wave packets in Chapter 2, we noted that there is a relationship between the *spread* of a function and its Fourier transform. When the de Broglie correspondence between wave number and momentum is made, the relationship takes the form

$$\Delta p \, \Delta x \geq \hbar$$

What we called the *spread* or, in the above context, the uncertainty can be sharpened mathematically to a definition: The uncertainty in any physical variable ΔA is equal to the *dispersion*, given by

$$(\Delta A)^2 \equiv \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 \tag{5A-1}$$

With this definition we can show that the uncertainty relation follows directly from quantum mechanics. Let us consider two hermitian operators A and B, and shift them by a constant, their expectation values in an arbitrary physical state $\psi(x)$, so that we have

$$U = A - \langle A \rangle \tag{5A-2}$$

and

$$V = B - \langle B \rangle \tag{5A-3}$$

where

$$\langle A \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) A \psi(x)$$

and so on. Given an arbitrary wave function $\psi(x)$, let us define

$$\phi(x) = (U + i\lambda V)\psi(x) \tag{5A-4}$$

with λ real. Whatever this function is, it will be true that

$$I(\lambda) = \int_{-\infty}^{\infty} dx \, \phi^*(x)\phi(x) \geq 0 \tag{5A-5}$$

This means, because of the hermiticity of U and V, that

$$\begin{aligned}
I(\lambda) &= \int_{-\infty}^{\infty} dx((U + i\lambda V)\psi(x))^*(U + i\lambda V)\psi(x) \\
&= \int_{-\infty}^{\infty} dx \, \psi^*(x)(U^+ - i\lambda V^+)(U + i\lambda V)\psi(x) \\
&= \int_{-\infty}^{\infty} dx \, \psi^*(x)(U - i\lambda V)(U + i\lambda V)\psi(x)
\end{aligned}$$

It follows that

$$I(\lambda) = \int_{-\infty}^{\infty} dx\, \psi^*(x)(U^2 + i\lambda[U, V] + \lambda^2 V^2)\psi(x)$$
$$= \langle U^2 \rangle + i\lambda\langle [U, V]\rangle + \lambda^2\langle V^2\rangle \tag{5A-6}$$

This will have its minimum value when

$$\frac{dI(\lambda)}{d\lambda} = 0 \tag{5A-7}$$

that is, when

$$i\langle [U, V]\rangle + 2\lambda\langle V^2\rangle = 0$$

When

$$\lambda_{\min} = -\frac{i\langle [U, V]\rangle}{2\langle V^2\rangle} \tag{5A-8}$$

is substituted into equation (5A-6) in the form

$$I(\lambda_{\min}) \geq 0 \tag{5A-9}$$

we get

$$\langle U^2\rangle\langle V^2\rangle \geq \frac{1}{4}\langle i[U, V]\rangle^2 \tag{5A-10}$$

or equivalently,

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\langle i[A, B]\rangle^2 \tag{5A-11}$$

For the operators p and x for which

$$[p, x] = -i\hbar \tag{5A-12}$$

this leads to

$$\Delta p\, \Delta x \geq \frac{\hbar}{2} \tag{5A-13}$$

Note that if for $\psi(x)$ we take an eigenstate of the operator A, for example, then

$$(\Delta A)^2 = \int_{-\infty}^{\infty} dx\, u_a^*(x)A^2 u_a(x) - \left(\int_{-\infty}^{\infty} dx\, u_a^*(x)Au_a(x)\right)^2$$
$$= a^2 \int_{-\infty}^{\infty} dx\, u_a^*(x)u_a(x) - \left(a\int_{-\infty}^{\infty} dx\, u_a^*(x)u_a(x)\right)^2 = 0$$

There is no problem, since the right side of the equation also vanishes:

$$\langle [A, B]\rangle = \int_{-\infty}^{\infty} dx\, u_a^*(x)(AB - BA)u_a(x) = \int_{-\infty}^{\infty} dx(Au_a(x))^*B - BA)u_a(x)$$
$$= a\int_{-\infty}^{\infty} dx\, u_a^*(x)Bu_a(x) - \int_{-\infty}^{\infty} dx\, u_a^*(x)Bau_a(x) = 0$$

We stress again that in this derivation no use was made of wave properties, x-space or p-space wave functions, or particle-wave duality. Our result depends entirely on the operator properties of the observables A and B.