Supplement 10-A

The Addition of Spin 1/2 and Orbital Angular Momentum (Details)

Of great importance for future applications is the combination of a spin with an orbital angular momentum. Since **L** depends on spatial coordinates and **S** does not, they commute

$$[\mathbf{L}, \mathbf{S}] = 0 \tag{10A-1}$$

It is therefore evident that the components of the total angular momentum **J**, defined by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{10A-2}$$

will satisfy the angular momentum commutation relations.

In asking for linear combinations of the Y_{lm} and the $\chi_\pm$ that are eigenstates of

$$J_z = L_z + S_z \tag{10A-3}$$

and

$$\begin{aligned} \mathbf{J}^2 &= \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}\cdot\mathbf{S} \\ &= \mathbf{L}^2 + \mathbf{S}^2 + 2L_zS_z + L_+S_- + L_-S_+ \end{aligned} \tag{10A-4}$$

we are again looking for the expansion coefficients of one complete set of eigenfunctions in terms of another set of eigenfunctions.

Let us consider the linear combination

$$\psi_{j,m+1/2} = \alpha Y_{lm}\chi_+ + \beta Y_{l,m+1}\chi_- \tag{10A-5}$$

It is, by construction, an eigenfunction of J_z with eigenvalue $(m + \frac{1}{2})\hbar$. We now determine α and β such that it is also an eigenfunction of J^2. We shall make use of the fact that

$$\begin{aligned} L_+Y_{lm} &= [l(l+1) - m(m+1)]^{1/2}\,\hbar Y_{l,m+1} \\ &= [(l+m+1)(l-m)]^{1/2}\,\hbar Y_{l,m+1} \\ L_-Y_{lm} &= [(l-m+1)(l+m)]^{1/2}\,\hbar Y_{l,m-1} \\ S_+\chi_+ &= S_-\chi_- = 0 \quad S_\pm\chi_\mp = \hbar\chi_\pm \end{aligned} \tag{10A-6}$$

Then

$$\begin{aligned} \mathbf{J}^2\psi_{j,m+1/2} = \alpha\hbar^2 \{&l(l+1)\,Y_{lm}\chi_+ + \tfrac{3}{4}Y_{lm}\chi_+ + 2m(\tfrac{1}{2})\,Y_{lm}\chi_+ \\ &+ [(l-m)(l+m+1)]^{1/2}\,Y_{l,m+1}\chi_-\} + \beta\hbar^2\{l(l+1)\,Y_{l,m+1}\chi_- \\ &+ \tfrac{3}{4}Y_{l,m+1}\chi_- + 2(m+1)(-\tfrac{1}{2})\,Y_{l,m+1}\chi_- \\ &+ [(l-m)(l+m+1)]^{1/2}\,Y_{lm}\chi_+\} \end{aligned} \tag{10A-7}$$

This will be of the form

$$\hbar^2 j(j+1)\,\psi_{j,m+1/2} = \hbar^2 j(j+1)(\alpha Y_{lm}\chi_+ + \beta Y_{l,m+1}\chi_-) \tag{10A-8}$$

provided that

$$\begin{aligned} \alpha[l(l+1) + \tfrac{3}{4} + m] + \beta[(l-m)(l+m+1)]^{1/2} &= j(j+1)\,\alpha \\ \beta[l(l+1) + \tfrac{3}{4} - m - 1] + \alpha[(l-m)(l+m+1)]^{1/2} &= j(j+1)\beta \end{aligned} \tag{10A-9}$$

This requires that

$$\begin{aligned} (l-m)(l+m+1) = {} & [j(j+1) - l(l+1) - \tfrac{3}{4} - m] \\ & \times [j(j+1) - l(l+1) - \tfrac{3}{4} + m + 1] \end{aligned}$$

which evidently has two solutions,

$$j(j+1) - l(l+1) - \tfrac{3}{4} = \begin{cases} -l-1 \\ l \end{cases} \tag{10A-10}$$

that is,

$$j = \begin{cases} l - \tfrac{1}{2} \\ l + \tfrac{1}{2} \end{cases} \tag{10A-11}$$

For $j = l + 1/2$, we get, after a little algebra

$$\alpha = \sqrt{\frac{l+m+1}{2l+1}} \qquad \beta = \sqrt{\frac{l-m}{2l+1}} \tag{10A-12}$$

(Actually we just get the ratio; these are already normalized forms.) Thus

$$\psi_{l+1/2,m+1/2} = \sqrt{\frac{l+m+1}{2l+1}}\,Y_{lm}\chi_+ + \sqrt{\frac{l-m}{2l+1}}\,Y_{l,m+1}\chi_- \tag{10A-13}$$

We can guess that the $j = l - 1/2$ solution must have the form

$$\psi_{l-1/2,m+1/2} = \sqrt{\frac{l-m}{2l+1}}\,Y_{lm}\chi_+ - \sqrt{\frac{l+m+1}{2l+1}}\,Y_{l,m+1}\chi_- \tag{10A-14}$$

in order to be orthogonal to the $j = l + 1/2$ solution.

General Rules for Addition of Angular Momenta, and Implications for Identical Particles

These two examples illustrate the general features that are involved in the addition of angular momenta: If we have the eigenstates $Y^{(1)}_{l_1m_1}$ of $\mathbf{L}_1^2$ and L_{1z}, and the eigenstates $Y^{(2)}_{l_2m_2}$ of $\mathbf{L}_2^2$ and L_{2z}, then we can form $(2l_1 + 1)(2l_2 + 1)$ product wave functions

$$Y^{(1)}_{l_1m_1}Y^{(2)}_{l_2m_2} \begin{Bmatrix} -l_1 \le m_1 \le l_1 \\ -l_2 \le m_2 \le l_2 \end{Bmatrix} \tag{10A-15}$$

These can be classified by the eigenvalue of

$$J_z = L_{1z} + L_{2z} \tag{10A-16}$$

which is $m_1 + m_2$, and which ranges from a maximum value of $l_1 + l_2$ down to $-l_1 - l_2$. As in the simple cases discussed earlier, different linear combinations of functions with the same m value will belong to different values of j. In the following table we list the possible combinations for the special example of $l_1 = 4$, $l_2 = 2$. We shall use the simple abbreviation (m_1, m_2) for $Y^{(1)}_{l_1m_1}Y^{(2)}_{l_2m_2}$.

m-value	m_1, m_2 combinations	numbers
6	(4, 2)	1
5	(4, 1) (3, 2)	2
4	(4, 0) (3, 1) (2, 2)	3
3	(4, −1) (3, 0) (2, 1) (1, 2)	4
2	(4, −2) (3, −1) (2, 0) (1, 1) (0, 2)	5
1	(3, −2) (2, −1) (1, 0) (0, 1) (−1, 2)	5
0	(2, −2) (1, −1) (0, 0) (−1, 1) (−2, 2)	5
−1	(1, −2) (0, −1) (−1, 0) (−2, 1) (−3, 2)	5
−2	(0, −2) (−1, −1) (−2, 0) (−3, 1) (−4, 2)	5
−3	(−1, −2) (−2, −1) (−3, 0) (−4, 1)	4
−4	(−2, −2) (−3, −1) (−4, 0)	3
−5	(−3, −2) (−4, −1)	2
−6	(−4, −2)	1

There are a total of 45 combinations, consistent with $(2l_1 + 1)(2l_2 + 1)$.

The highest state has total angular momentum $l_1 + l_2$ as can easily be checked by applying J^2 to $Y^{(1)}_{l_1l_1}Y^{(2)}_{l_2l_2}$:

$$\begin{aligned}\mathbf{J}^2 Y^{(1)}_{l_1l_1}Y^{(2)}_{l_2l_2} &= (\mathbf{L}_1^2 + \mathbf{L}_2^2 + 2L_{1z}L_{2z} + L_{1+}L_{2-} + L_{1-}L_{2+})\, Y^{(1)}_{l_1l_1}Y^{(2)}_{l_2l_2} \\ &= \hbar^2[l_1(l_1+1) + l_2(l_2+1) + 2l_1l_2]\, Y^{(1)}_{l_1l_1}Y^{(2)}_{l_2l_2} \qquad (10A\text{-}17) \\ &= \hbar^2(l_1+l_2)(l_1+l_2+1)\, Y^{(1)}_{l_1l_1}Y^{(2)}_{l_2l_2}\end{aligned}$$

This is $j = 6$ in the example discussed in the table. Successive applications of

$$J_- = L_{1-} + L_{2-} \qquad (10A\text{-}18)$$

will pick out one linear combination from each row in the table. These will form the 13 states that belong to $j = 6$. When this is done, there remains a single state with $m = 5$, two with $m = 4, \dots,$ one with $m = -5$. It is extremely plausible, and can, in fact, be checked, that the $m = 5$ state belongs to $j = 5$. Again, successive applications of J_- pick out another linear combination from each row in the table, forming 11 states that belong to $j = 5$. Repetition of this procedure shows that we get, after this, sets that belong to $j = 4$, $j = 3$, and finally $j = 2$. The multiplicities add up to 45:

$$13 + 11 + 9 + 7 + 5 = 45$$

We shall not work out the details of this decomposition, as it is beyond the scope of this book. We merely state the results.

(a) The produces $Y^{(1)}_{l_1m_1}Y^{(2)}_{l_2m_2}$ can be decomposed into eigenstates of $\mathbf{J}^2$, with eigenvalues $j(j+1)\hbar^2$, where j can take on the values

$$j = l_1 + l_2, l_1 + l_2 - 1, \dots, |l_1 - l_2| \qquad (10A\text{-}19)$$

We can verify that the multiplicities check in (10A-19): If we sum the number of states, we get ($l_1 \geq l_2$)

$$\begin{aligned}
[2(l_1 + l_2) + 1] + [2(l_1 + l_2 - 1) + 1] + \cdots + [2(l_1 - l_2) + 1]& \\
= \sum_{n=0}^{2l_2} [2(l_1 - l_2 + n) + 1]& \\
= (2l_2 + 1)(2l_1 + 1)&
\end{aligned} \tag{10A-20}$$

(b) It is possible to generatize (10A-13) and (10A-14) to give the Clebsch-Gordan series

$$\psi_{jm} = \sum_{m1} C(jm;\, l_1m_1l_2m_2)\, Y^{(1)}_{l_1m_1} Y^{(2)}_{l_2m_2} \tag{10A-21}$$

The coefficients $C(jm;\, l_1m_1m_2)$ are called Clebsch-Gordan coefficients, and they have been tabulated for many values of the arguments. We calculated the coefficients for $l_2 = 1/2$, and summarize (10A-12) and (10A-13) in the table that follows. Note that $m = m_1 + m_2$, so that the m in (10A-13) and (10A-14) is really m_1 below.

$C(jm;\, l_1m_1,\, 1/2,\, m_2$

	$m_2 = 1/2$	$m_2 = -1/2$
$j = l_1 + 1/2$	$\sqrt{\dfrac{l_1 + m + 1/2}{2l_1 + 1}}$	$\sqrt{\dfrac{l_1 - m + 1/2}{2l_1 + 1}}$
$j = l_1 - 1/2$	$-\sqrt{\dfrac{l_1 - m + 1/2}{2l_1 + 1}}$	$\sqrt{\dfrac{l_1 + m + 1/2}{2l_1 + 1}}$

Another useful table is

$C(jm;\, l_1m_1,\, 1,\, m_2)$

	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j = l_1 + 1$	$\sqrt{\dfrac{(l_1 + m)(l_1 + m + 1)}{(2l_1 + 1)(2l_1 + 2)}}$	$\sqrt{\dfrac{(l_1 - m + 1)(l_1 + m + 1)}{(2l_1 + 1)(l_1 + 1)}}$	$\sqrt{\dfrac{(l_1 - m)(l_1 - m + 1)}{(2l_1 + 1)(2l_1 + 2)}}$
$j = l_1$	$-\sqrt{\dfrac{(l + m)(l_1 - m + 1)}{2l_1(l_1 + 1)}}$	$\dfrac{m}{\sqrt{l_1(l_1 + 1)}}$	$\sqrt{\dfrac{(l_1 - m)(l_1 + m + 1)}{2l_1(2l_1 + 1)}}$
$j = l_1 - 1$	$\sqrt{\dfrac{(l_1 - m)(l_1 - m + 1)}{2l_1(2l_1 + 1)}}$	$-\sqrt{\dfrac{(l_1 - m)(l_1 + m)}{l_1(2l_1 + 1)}}$	$\sqrt{\dfrac{(l_1 + m)(l_1 + m + 1)}{2l_1(2l_1 + 1)}}$

Supplement 10-B

The Levi-Civita Symbol and Maxwell's Equations

A very useful mathematical device is the use of the *Levi-Civita* symbol. The symbol e_{ijk} is defined by the following properties:

(a) It is antisymmetric under the interchange of any two of its indices. For example,

$$e_{123} = -e_{213} = -(-e_{231}) \tag{10B-1}$$

and so on. Two consequences of this rule are

(i) When any two indices are equal, the value of e_{ijk} is *zero*.

(ii) $e_{123} = e_{231} = e_{312}$

(b) $e_{123} = 1$ (10B-2)

Some consequences of this definition are

$$\begin{aligned} e_{ijk}e_{ijm} &= 2\delta_{km} \\ e_{ijk}e_{imn} &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \\ e_{ijk}A_jB_k &= (A \times B)_i \\ [L_i, L_j] &= ie_{ijk}L_k \end{aligned} \tag{10B-3}$$

We may use this to write out Maxwell's equations in a particularly interesting way.

Maxwell's equations in empty space have the form

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{10B-4}$$

They may be rewritten in the form

$$\frac{\partial}{\partial t}(\mathbf{E} + ic\mathbf{B}) = -ic\nabla \times (\mathbf{E} + ic\mathbf{B}) \tag{10B-5}$$

Which bears some resemblance to the Schrödinger equation in that the "wave function" is complex, and that the first-order time derivative enters into the equation.

We may write the equation in a very suggestive way by using the Levi-Civita symbol in two contexts. First, the symbol may be used to give a matrix representation of the spin 1 angular momentum **S**. (We are actually working with the angular momentum matrix divided by $\hbar$—that is, with the analog of $\boldsymbol{\sigma}/2$.)

To see this, we postulate

$$(S_i)_{jk} = -ie_{ijk} \tag{10B-6}$$

The square of the matrix is easily calculated. We have

$$(\mathbf{S}^2)_{jl} = (\mathbf{S}_i)_{jk}(\mathbf{S}_i)_{kl} = -e_{ijk}e_{ikl} = e_{ijk}e_{ikl} = 2\delta_{jl} \tag{10B-7}$$

We next need to check the commutation relations

$$\begin{aligned}(S_a)_{jk}(S_b)_{kl} - (S_b)_{jk}(S_a)_{kl} &= -e_{ajk}e_{bkl} + e_{bjk}e_{akl} \\ &= e_{ajk}e_{blk} - e_{bjk}e_{alk} = \delta_{ab}\delta_{jl} - \delta_{al}\delta_{jb} - \delta_{ab}\delta_{jl} + \delta_{bl}\delta_{aj} \\ &= \delta_{bl}\delta_{aj} - \delta_{al}\delta_{jb} = e_{bam}e_{ljm} = ie_{abm}(-ie_{mjl}) = ie_{abm}(S_m)_{jl}\end{aligned}$$

that is,

$$[S_a, S_b] = ie_{abm}S_m \tag{10B-8}$$

Let us now rewrite our version of Maxwell's equations. It reads

$$\begin{aligned}\frac{\partial}{\partial t}(E_i + icB_i) &= -ice_{imn}\frac{\partial}{\partial x_m}(E_n + iB_n) \\ &= -c(S_m)_{in}\frac{\partial}{\partial x_m}(E_n + iB_n)\end{aligned}$$

or equivalently,

$$i\hbar\frac{\partial}{\partial t}(E_i + icB_i) = c(S_m)_{in}\frac{\hbar}{i}\frac{\partial}{\partial x_m}(E_n + iB_n) \tag{10B-9}$$

With the notation $\psi_i = (E_i + icB_i)$, we get

$$i\hbar\frac{\partial\psi_i}{\partial t} = c(\mathbf{S}\cdot\mathbf{p}_{\text{op}})_{im}\,\psi_m \tag{10B-10}$$

The operator on the right side is the projection of the photon spin along the direction of motion. The complex conjugate wave function is easily seen to satisfy

$$i\hbar\frac{\partial\psi_i^*}{\partial t} = -c(\mathbf{S}\cdot\mathbf{p}_{\text{op}})_{im}\,\psi_m^* \tag{10B-11}$$

where the right side represents the opposite projection (helicity). We need both equations to obtain separate equations for **E** and **B**.