The Addition of Spin 1/2 and Orbital Angular Momentum (Details)

Of great importance for future applications is the combination of a spin with an orbital angular momentum. Since $\mathbf{L}$ depends on spatial coordinates and $\mathbf{S}$ does not, they commute

$$[\mathbf{L}, \mathbf{S}] = 0 \quad (10A-1)$$

It is therefore evident that the components of the total angular momentum $\mathbf{J}$, defined by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \quad (10A-2)$$

will satisfy the angular momentum commutation relations.

In asking for linear combinations of the $Y_{lm}$ and the $\chi_{\pm}$ that are eigenstates of

$$J_z = L_z + S_z \quad (10A-3)$$

and

$$\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L} \cdot \mathbf{S} \quad (10A-4)$$

we are again looking for the expansion coefficients of one complete set of eigenfunctions in terms of another set of eigenfunctions.

Let us consider the linear combination

$$\psi_{j,m+\frac{1}{2}} = \alpha Y_{lm} \chi_+ + \beta Y_{l,m+1} \chi_- \quad (10A-5)$$

It is, by construction, an eigenfunction of $J_z$ with eigenvalue $(m + \frac{1}{2})\hbar$. We now determine $\alpha$ and $\beta$ such that it is also an eigenfunction of $\mathbf{J}^2$. We shall make use of the fact that

$$L_+ Y_{lm} = [(l + 1)(l + m) - m(m + 1)]^{1/2} \hbar Y_{lm+1} \quad (10A-6)$$

$$L_- Y_{lm} = [(l + m + 1)(l + 1) - (l - m)]^{1/2} \hbar Y_{lm-1}$$

$$S_+ \chi_+ = S_- \chi_- = 0 \quad S_+ \chi_\mp = \hbar \chi_\mp$$

Then

$$\mathbf{J}^2 \psi_{j,m+\frac{1}{2}} = \hbar^2 \left\{ \frac{1}{2} (l + 1) Y_{lm} \chi_+ + \frac{3}{4} Y_{lm} \chi_+ + \frac{2m+1}{2} Y_{lm} \chi_+ + [(l-m)(l+m+1)]^{1/2} Y_{l,m+1} \chi_- \right\} + \frac{1}{2} \beta \hbar^2 \left\{ \frac{3}{2} Y_{l,m+1} \chi_- + 2(l+1)(-\frac{1}{2}) Y_{l,m+1} \chi_- + [(l-m)(l+m+1)]^{1/2} Y_{lm} \chi_+ \right\} \quad (10A-7)$$
This will be of the form
\[ \hbar^2 j(j + 1) \psi_{j,m+1/2} = \hbar^2 j(j + 1)(\alpha Y_{lm}X_+ + \beta Y_{l,m+1}X_-) \]  
(10A-8)
provided that
\[ \alpha[(l + 1) + \frac{3}{2} + m] + \beta[(l - m)(l + m + 1)]^{1/2} = j(j + 1) \alpha \]
\[ \beta[(l + 1) + \frac{3}{2} - m] + \alpha[(l - m)(l + m + 1)]^{1/2} = j(j + 1) \beta \]  
(10A-9)
This requires that
\[ (l - m)(l + m + 1) = [j(j + 1) - l(l + 1) - \frac{3}{2} - m] \times [j(j + 1) - l(l + 1) - \frac{3}{2} + m + 1] \]
which evidently has two solutions,
\[ j(j + 1) - l(l + 1) - \frac{3}{2} = \begin{cases} -l - 1 \\ l \end{cases} \]  
(10A-10)
that is,
\[ j = \begin{cases} l - \frac{1}{2} \\ l + \frac{1}{2} \end{cases} \]  
(10A-11)
For \( j = l + 1/2 \), we get, after a little algebra
\[ \alpha = \sqrt{\frac{l + m + 1}{2l + 1}} \quad \beta = \sqrt{\frac{l - m}{2l + 1}} \]  
(10A-12)
(Actually we just get the ratio; these are already normalized forms.) Thus
\[ \psi_{l+1/2,m+1/2} = \sqrt{\frac{l + m + 1}{2l + 1}} Y_{lm}X_+ + \sqrt{\frac{l - m}{2l + 1}} Y_{l,m+1}X_- \]  
(10A-13)
We can guess that the \( j = l - 1/2 \) solution must have the form
\[ \psi_{l-1/2,m+1/2} = \sqrt{\frac{l - m}{2l + 1}} Y_{lm}X_+ - \sqrt{\frac{l + m + 1}{2l + 1}} Y_{l,m+1}X_- \]  
(10A-14)
in order to be orthogonal to the \( j = l + 1/2 \) solution.

**General Rules for Addition of Angular Momenta, and Implications for Identical Particles**

These two examples illustrate the general features that are involved in the addition of angular momenta: If we have the eigenstates \( Y_{lm}^{(1)} \) of \( L_1^2 \) and \( L_{1z} \), and the eigenstates \( Y_{l,m}^{(2)} \) of \( L_2^2 \) and \( L_{2z} \), then we can form \((2l_1 + 1)(2l_2 + 1)\) product wave functions
\[ Y_{l,m_1}^{(1)} Y_{l,m_2}^{(2)} \begin{cases} -l_1 \leq m_1 \leq l_1 \\ -l_2 \leq m_2 \leq l_2 \end{cases} \]  
(10A-15)
These can be classified by the eigenvalue of
\[ J_z = L_{1z} + L_{2z} \]  
(10A-16)
which is \( m_1 + m_2 \), and which ranges from a maximum value of \( l_1 + l_2 \) down to \(-l_1 - l_2\). As in the simple cases discussed earlier, different linear combinations of functions with
the same \( m \) value will belong to different values of \( j \). In the following table we list the possible combinations for the special example of \( l_1 = 4, l_2 = 2 \). We shall use the simple abbreviation \((m_1, m_2)\) for \( Y_{l_1 m_1}^{(1)} Y_{l_2 m_2}^{(2)}\).

<table>
<thead>
<tr>
<th>( m )-value</th>
<th>( m_1, m_2 ) combinations</th>
<th>numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>(4, 2)</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>(4, 1) (3, 2)</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>(4, 0) (3, 1) (2, 2)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>(4, –1) (3, 0) (2, 1) (1, 2)</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>(4, –2) (3, –1) (2, 0) (1, 1) (0, 2)</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>(3, –2) (2, –1) (1, 0) (0, 1) (–1, 2)</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>(2, –2) (1, –1) (0, 0) (–1, 1) (–2, 2)</td>
<td>5</td>
</tr>
<tr>
<td>–1</td>
<td>(1, –2) (0, –1) (–1, 0) (–2, 1) (–3, 2)</td>
<td>5</td>
</tr>
<tr>
<td>–2</td>
<td>(0, –2) (–1, –1) (–2, 0) (–3, 1) (–4, 2)</td>
<td>5</td>
</tr>
<tr>
<td>–3</td>
<td>(–1, –2) (–2, –1) (–3, 0) (–4, 1)</td>
<td>4</td>
</tr>
<tr>
<td>–4</td>
<td>(–2, –2) (–3, –1) (–4, 0)</td>
<td>3</td>
</tr>
<tr>
<td>–5</td>
<td>(–3, –2) (–4, –1)</td>
<td>2</td>
</tr>
<tr>
<td>–6</td>
<td>(–4, –2)</td>
<td>1</td>
</tr>
</tbody>
</table>

There are a total of 45 combinations, consistent with \((2l_1 + 1) (2l_2 + 1)\).

The highest state has total angular momentum \( l_1 + l_2 \) as can easily be checked by applying \( \mathbf{J}^2 \) to \( Y_{l_1 l_2}^{(1)} Y_{l_1 l_2}^{(2)}\):

\[
\mathbf{J}^2 Y_{l_1 l_2}^{(1)} Y_{l_1 l_2}^{(2)} = (\mathbf{L}_1^2 + \mathbf{L}_2^2 + 2L_1 L_2 + L_{1+} L_{2-} + L_{1-} L_{2+}) Y_{l_1 l_2}^{(1)} Y_{l_1 l_2}^{(2)}
\]

\[
= \hbar^2 (l_1 (l_1 + 1) + l_2 (l_2 + 1) + 2l_1 l_2) Y_{l_1 l_2}^{(1)} Y_{l_1 l_2}^{(2)}
\]

\[
= \hbar^2 (l_1 + l_2)(l_1 + l_2 + 1) Y_{l_1 l_2}^{(1)} Y_{l_1 l_2}^{(2)} \quad (10A-17)
\]

This is \( j = 6 \) in the example discussed in the table. Successive applications of

\[
J_+ = L_{1+} + L_{2-}
\]

will pick out one linear combination from each row in the table. These will form the 13 states that belong to \( j = 6 \). When this is done, there remains a single state with \( m = 5 \), two with \( m = 4, \ldots \), one with \( m = –5 \). It is extremely plausible, and can, in fact, be checked, that the \( m = 5 \) state belongs to \( j = 5 \). Again, successive applications of \( J_- \) pick out another linear combination from each row in the table, forming 11 states that belong to \( j = 5 \). Repetition of this procedure shows that we get, after this, sets that belong to \( j = 4, j = 3, \) and finally \( j = 2 \). The multiplicities add up to 45:

\[
13 + 11 + 9 + 7 + 5 = 45
\]

We shall not work out the details of this decomposition, as it is beyond the scope of this book. We merely state the results.

(a) The produces \( Y_{l_1 m_1}^{(1)} Y_{l_2 m_2}^{(2)} \) can be decomposed into eigenstates of \( \mathbf{J}^2 \), with eigenvalues \( j(j + 1) \hbar^2 \), where \( j \) can take on the values

\[
j = l_1 + l_2, l_1 + l_2 - 1, \ldots, |l_1 - l_2|
\]

\[
(10A-19)
\]
We can verify that the multiplicities check in (10A-19): If we sum the number of states, we get \( l_1 \geq l_2 \)

\[
(2l_1 + l_2 + 1) + 2(l_1 + l_2 - 1) + \cdots + 2(l_1 - l_2 + 1) = \sum_{n=0}^{2l_1} [2(l_1 - l_2 + n) + 1] = (2l_2 + 1)(2l_1 + 1) \tag{10A-20}
\]

(b) It is possible to generate (10A-13) and (10A-14) to give the Clebsch-Gordan

\[
\psi_{jm} = \sum_{m_1} C(jm; l_1m_1l_2m_2) Y_{l_1m_1}^{(1)} Y_{l_2m_2}^{(2)} \tag{10A-21}
\]

The coefficients \( C(jm; l_1m_1l_2m_2) \) are called Clebsch-Gordan coefficients, and they have been tabulated for many values of the arguments. We calculated the coefficients for \( l_2 = 1/2 \), and summarize (10A-12) and (10A-13) in the table that follows. Note that \( m = m_1 + m_2 \), so that the \( m \) in (10A-13) and (10A-14) is really \( m_1 \) below.

<table>
<thead>
<tr>
<th>( l_1 )</th>
<th>( m_2 = 1/2 )</th>
<th>( m_2 = -1/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = l_1 + 1/2 )</td>
<td>( \sqrt{\frac{l_1 + m + 1/2}{2l_1 + 1}} )</td>
<td>( \sqrt{\frac{l_1 - m + 1/2}{2l_1 + 1}} )</td>
</tr>
<tr>
<td>( j = l_1 - 1/2 )</td>
<td>( -\sqrt{\frac{l_1 - m + 1/2}{2l_1 + 1}} )</td>
<td>( \sqrt{\frac{l_1 + m + 1/2}{2l_1 + 1}} )</td>
</tr>
</tbody>
</table>

Another useful table is

<table>
<thead>
<tr>
<th>( l_1 )</th>
<th>( m_2 = 1 )</th>
<th>( m_2 = 0 )</th>
<th>( m_2 = -1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = l_1 + 1 )</td>
<td>( \sqrt{\frac{(l_1 + m)(l_1 + m + 1)}{(2l_1 + 1)(2l_1 + 2)}} )</td>
<td>( \sqrt{\frac{(l_1 - m + 1)(l_1 + m + 1)}{(2l_1 + 1)(2l_1 + 2)}} )</td>
<td>( \sqrt{\frac{(l_1 - m)(l_1 - m + 1)}{(2l_1 + 1)(2l_1 + 2)}} )</td>
</tr>
<tr>
<td>( j = l_1 )</td>
<td>( -\sqrt{\frac{(l_1 + m)(l_1 - m + 1)}{2l_1(l_1 + 1)}} )</td>
<td>( \frac{m}{\sqrt{l_1(l_1 + 1)}} )</td>
<td>( \sqrt{\frac{(l_1 - m)(l_1 + m)}{2l_1(l_1 + 1)}} )</td>
</tr>
<tr>
<td>( j = l_1 - 1 )</td>
<td>( \sqrt{\frac{(l_1 - m)(l_1 - m + 1)}{2l_1(2l_1 + 1)}} )</td>
<td>( -\sqrt{\frac{(l_1 - m)(l_1 + m)}{l_1(2l_1 + 1)}} )</td>
<td>( \sqrt{\frac{(l_1 + m)(l_1 + m + 1)}{2l_1(2l_1 + 1)}} )</td>
</tr>
</tbody>
</table>
The Levi-Civita Symbol and Maxwell’s Equations

A very useful mathematical device is the use of the Levi-Civita symbol. The symbol $e_{ijk}$ is defined by the following properties:

(a) It is antisymmetric under the interchange of any two of its indices. For example,

$$e_{123} = -e_{213} = -(e_{321})$$  \hspace{2cm} (10B-1)

and so on. Two consequences of this rule are

(i) When any two indices are equal, the value of $e_{ijk}$ is zero.

(ii) $e_{123} = e_{231} = e_{312}$

(b) $e_{123} = 1$  \hspace{2cm} (10B-2)

Some consequences of this definition are

$$e_{ijk}e_{ilm} = 2\delta_{km}$$

$$e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$e_{ijk}A_{l}B_{k} = (A \times B)_{i}$$

$$[L_{i}, L_{j}] = ie_{ijk}L_{k}$$  \hspace{2cm} (10B-3)

We may use this to write out Maxwell’s equations in a particularly interesting way.

Maxwell’s equations in empty space have the form

$$\nabla \cdot B = 0$$

$$\nabla \cdot E = 0$$

$$\nabla \times B = \frac{1}{c^{2}} \frac{\partial E}{\partial t}$$  \hspace{2cm} (10B-4)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

They may be rewritten in the form

$$\frac{\partial}{\partial t} (E + icB) = -ic \nabla \times (E + icB)$$  \hspace{2cm} (10B-5)

Which bears some resemblance to the Schrödinger equation in that the “wave function” is complex, and that the first-order time derivative enters into the equation.

We may write the equation in a very suggestive way by using the Levi-Civita symbol in two contexts. First, the symbol may be used to give a matrix representation of the spin 1 angular momentum $S$. (We are actually working with the angular momentum matrix divided by $\hbar$—that is, with the analog of $\sigma/2$.)
To see this, we postulate

$$(S)_{jk} = -ie_{ijk}$$  \hspace{1cm} (10B-6)$$

The square of the matrix is easily calculated. We have

$$(S^2)_{jl} = (S)_{jk}(S)_{kl} = -e_{ijk} e_{ikl} = e_{ijk} e_{ikl} = 2\delta_{jl}$$  \hspace{1cm} (10B-7)$$

We next need to check the commutation relations

$$((S_a)_{jk}(S_b)_{kl} - (S_b)_{jk}(S_a)_{kl}) = -e_{ajk} e_{bkl} + e_{bjk} e_{akl}$$

$$= e_{ajk} e_{bkl} - e_{bjk} e_{akl} = \delta_{al}\delta_{jl} - \delta_{aj}\delta_{jl} + \delta_{bj}\delta_{al}$$

that is,

$$[S_a, S_b] = ie_{a0m} S_m$$  \hspace{1cm} (10B-8)$$

Let us now rewrite our version of Maxwell’s equations. It reads

$$\frac{\partial}{\partial t} (E_i + icB_i) = -iec_{inn} \frac{\partial}{\partial x_m} (E_n + iB_n)$$

$$= -c(S_m)_{in} \frac{\partial}{\partial x_m} (E_n + iB_n)$$

or equivalently,

$$i\hbar \frac{\partial}{\partial t} (E_i + icB_i) = c(S_m)_{in} \frac{\hbar}{i} \frac{\partial}{\partial x_m} (E_n + iB_n)$$  \hspace{1cm} (10B-9)$$

With the notation $\psi_i = (E_i + icB_i)$, we get

$$i\hbar \frac{\partial \psi_i}{\partial t} = c(S \cdot p_{\psi_i})_{im} \psi_m$$  \hspace{1cm} (10B-10)$$

The operator on the right side is the projection of the photon spin along the direction of motion. The complex conjugate wave function is easily seen to satisfy

$$i\hbar \frac{\partial \psi_i^*}{\partial t} = -c(S \cdot p_{\psi_i})_{im} \psi_m^*$$  \hspace{1cm} (10B-11)$$

where the right side represents the opposite projection (helicity). We need both equations to obtain separate equations for $E$ and $B$. 