Supplement 13-A

Conservation of Total Momentum

In our discussion of angular momentum in Chapter 8 we found that the assumption of invariance of the Hamiltonian under rotations led to the appearance of a new constant of motion, *the angular momentum*. In this supplement we show that the assumption of *invariance under spatial displacement* leads to the existence of a constant of the motion, *the momentum*. The requirement that the system be unchanged under the transformation

$$x_i \rightarrow x_i + a \tag{13A-1}$$

does not change the kinetic energy, since a is independent of time. The potential energy will change, *unless* it has the form

$$V(x_1, x_2, x_3, \ldots, x_N) = V(x_1 - x_2, x_1 - x_3 \ldots x_2 - x_3, \ldots, x_{N-1} - x_N) \tag{13A-2}$$

In classical mechanics, the absence of external forces leads to momentum conservation. This follows from the equations of motion,

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial}{\partial x_i} V(x_1 - x_2, x_1 - x_3, \ldots, x_{N-1} - x_N) \tag{13A-3}$$

a consequence of which is that

$$\begin{aligned} \frac{d}{dt} \sum_i m_i \frac{dx_i}{dt} &= -\sum_i \frac{\partial}{\partial x_i} V(x_1 - x_2, \ldots, x_{N-1} - x_N) \\ &= 0 \end{aligned} \tag{13A-4}$$

The reason for the vanishing of the right side of the preceding equation is that for every argument in V, there are equal and opposite contributions that come from $\sum_i \partial/\partial x_i$ acting on it. For example, with $u = x_1 - x_2$, $v = x_1 - x_3$, $w = x_2 - x_3$,

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right) V(u, v, w) = \frac{\partial V}{\partial u} + \frac{\partial V}{\partial v} - \frac{\partial V}{\partial u} + \frac{\partial V}{\partial w} - \frac{\partial V}{\partial v} - \frac{\partial V}{\partial w} = 0$$

Hence

$$P = \sum_i m_i \frac{dx_i}{dt} \tag{13A-5}$$

is a constant of the motion.

In quantum mechanics the same conclusion holds. We shall demonstrate it by using the invariance of the Hamiltonian under the transformation (13A-1). The invariance implies that both

$$Hu_E(x_1, x_2, \ldots, x_N) = Eu_E(x_1, x_2, \ldots, x_N) \tag{13A-6}$$

and

$$Hu_E(x_1 + a, x_2 + a, \ldots, x_N + a) = Eu_E(x_1 + a, x_2 + a, \ldots, X_N + a) \tag{13A-7}$$

hold. Let us take a infinitesimal, so that terms of $0(a^2)$ can be neglected. Then

$$\begin{aligned} u(x_1 + a, \ldots, x_N + a) &\simeq u(x_1, \ldots, x_N) + a \frac{\partial}{\partial x_1} u(x_1, \ldots, x_N) \\ &\quad + a \frac{\partial}{\partial x_2} u(x_1, \ldots, x_N) + \cdots \\ &\simeq u(x_1, \ldots, x_N) + a \sum_i \frac{\partial}{\partial x_i} u(x_1, \ldots, x_N) \end{aligned} \tag{13A-8}$$

and hence

$$\begin{aligned} aH\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\right) u_E(x_1, \ldots, x_N) &= aE\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\right) u_E(x_1, \ldots, x_N) \\ &= a\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\right) E u_E(x_1, \ldots, x_N) \\ &= a\left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\right) H u_E(x_1, \ldots, x_N) \end{aligned} \tag{13A-9}$$

If we now define

$$P = \frac{\hbar}{i} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \equiv \sum_{i=1}^{N} p_i \tag{13A-10}$$

so that P is the total momentum operator, we see that we have demonstrated that

$$(HP - PH)\, u_E(x_1, \ldots, x_N) = 0 \tag{13A-11}$$

Since the energy eigenstates for N-particles form a complete set of states, in the sense that any function of $x_1, x_2, \ldots, x_N$ can be expanded in terms of all the $u_E(x_1, \ldots, x_N)$, the preceding equation can be translated into

$$[H, P]\psi(x_1, \ldots, x_N) = 0 \tag{13A-12}$$

for all $\psi(x_1, \ldots, x_N)$—that is, into the operator relation

$$[H, P] = 0 \tag{13A-13}$$

This, however, implies that P, the total momentum of the system, is a *constant of the motion*. This is a very deep consequence of what is really a statement about the nature of space. The statement that there is no origin—that is, that the laws of physics are invariant under displacement by a fixed distance—leads to a conservation law. In relativistic quantum mechanics there are no potentials of the form that we consider here; nevertheless the invariance principle, as stated earlier, still leads to a conserved total momentum.