Supplement 16-A

The Aharanov–Bohm Effect

Let us return to the description of an electron of charge $-e$ and mass m_e, in a time-independent magnetic field. The system is described by the Hamiltonian

$$H = \frac{1}{2m_e}\left(-i\hbar\,\nabla + e\mathbf{A}(\mathbf{r})\right)^2 - e\phi(\mathbf{r}) \tag{16A-1}$$

and the time-independent Schrödinger equation reads

$$H\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{16A-2}$$

In the absence of a vector potential the Hamiltonian reads

$$H_0 = \frac{1}{2m_e}\left(-i\hbar\,\nabla\right)^2 - e\phi(\mathbf{r}) \tag{16A-3}$$

and the Schrödinger equation reads

$$H_0\psi_0(\mathbf{r}) = E\psi_0(\mathbf{r}) \tag{16A-4}$$

We can show that *formally* the solution of (16A-4) and (16A-2) are related by a simple phase factor. Let us write a general expression

$$\psi(\mathbf{r}) = e^{ie\Lambda(\mathbf{r})}\,\psi_0(\mathbf{r}) \tag{16A-5}$$

It follows that

$$(-i\hbar\,\nabla + e\mathbf{A})\,\psi = e\hbar\,\nabla\Lambda e^{ie\Lambda}\,\psi_0 + e^{ie\Lambda}\,(-i\hbar\,\nabla + e\mathbf{A})\,\psi_0 \tag{16A-6}$$

We now observe that if we choose Λ such that

$$e\hbar\,\nabla\Lambda + e\mathbf{A} = 0 \tag{16A-7}$$

then

$$(-i\hbar\,\nabla + e\mathbf{A})\,\psi = e^{ie\Lambda}\,(-i\hbar\,\nabla)\,\psi_0 \tag{16A-8}$$

When this is repeated, we obtain

$$H\psi = e^{ie\Lambda}H_0\psi_0 = e^{ie\Lambda}\,E\psi_0 = E\psi \tag{16A-9}$$

What we have just shown is that we can relate a solution of the Schrodinger equation with a vector potential $\mathbf{A}(\mathbf{r})$ to one of the Schrodinger equation *without* a vector potential by performing the *gauge transformation* shown in Eq. (16A-5). This depended, however, on our ability for find a gauge function $\Lambda(\mathbf{r})$ that satisfies Eq. (16A-7). This can only happen if

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = -\hbar\nabla \times \nabla\Lambda(\mathbf{r}) = 0$$

You may wonder why we bother to go through this, since if $\mathbf{B} = 0$, we did not have to write out the Schrodinger equation in the form (16A-1). The point is that in quantum mechanics it is the potentials that matter, and there are circumstances where their presence creates physical

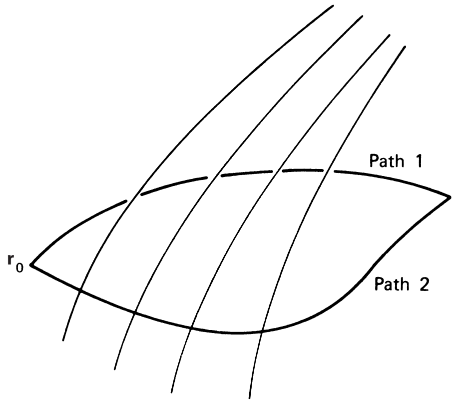

Figure 16A-1 The integrals $\int_{r_0}^{r} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'$ along path 1 and path 2 are generally not the same, since the difference is equal to the magnetic flux Φ enclosed by the closed loop.

effects even when the magnetic field vanishes. We can write out the dependence on the vector potential by noting that Eq. (16A-7) can be solved by a *line integral*,

$$\Lambda(\mathbf{r}) = -\frac{1}{\hbar}\int_P^r d\mathbf{l}' \cdot \mathbf{A}(\mathbf{r}')$$

which starts at some fixed point P and goes to $\mathbf{r}$. In terms of this (16A-5) takes the form

$$\psi(\mathbf{r}) = e^{-i(e/\hbar)\int_P^r d\ell' \cdot \mathbf{A}(\mathbf{r}')}\psi_0(\mathbf{r}) \tag{16A-10}$$

Is the line integral in the phase is independent of the path taken between P and $\mathbf{r}$? Let us consider two paths as shown in Fig. 16A-1. The difference between the line integrals is

$$\int_2 d\mathbf{l} \cdot \mathbf{A} - \int_1 d\mathbf{l} \cdot \mathbf{A} = \oint_{counterclockwise} d\mathbf{l} \cdot \mathbf{A} \tag{16A-11}$$

By Stokes' theorem we have

$$\oint d\mathbf{l} \cdot \mathbf{A} = \iint_{encl.\ surface} d\mathbf{S} \cdot \nabla \times \mathbf{A} = \iint_S d\mathbf{S} \cdot \mathbf{B} = \Phi \tag{16A-12}$$

This shows that for all paths that do not enclose any magnetic flux, the integral in the phase factor is the same, so that the phase factor does not depend on the path taken. The important point is that there may be situations when there is a local magnetic field, and paths that go around the flux tube are not equivalent.

In 1959 Y. Aharanov and D. Bohm pointed out a previously overlooked[1] aspect of the quantum mechanics of a charged particle in the presence of electromagnetic fields—namely, that even in a field free region, in which $\mathbf{B} = 0$, the presence in a field elsewhere can have physical consequences because $\mathbf{A} \neq 0$.

Consider, for example, a two-slit diffraction experiment carried out with electrons, as shown in Fig. 16A-2. Suppose that there is a solenoid perpendicular to the plane in which the electrons move, located as in Fig. 16A-2(*a*) away from the slits. The interference pattern at the screen depends on the difference in phase between the wave functions for the

[1]Actually the A-B effect was discovered in an earlier paper by W. Ehrenberg and R. E. Siday (1949). Since the effects of potentials were not of central interest in the paper, their important remarks on this subject were completely overlooked.

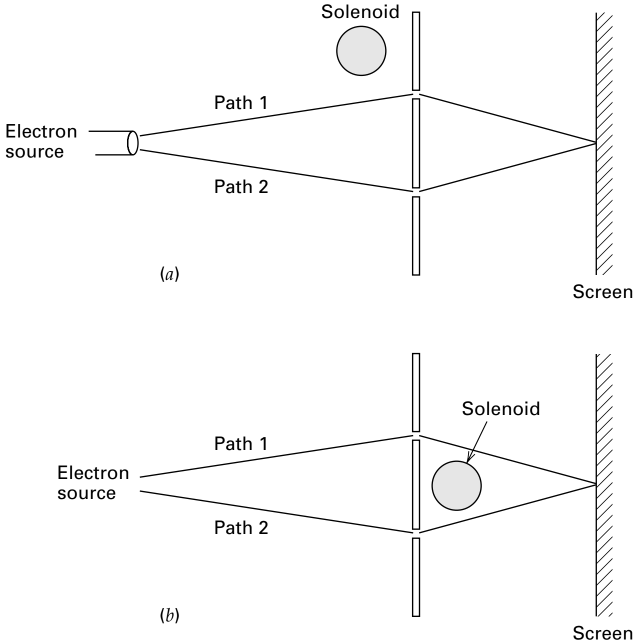

Figure 16A-2 Schematic sketch of experiment measuring shift of electron interference pattern by confined magnetic flux.

electrons passing through the two slits. In the absence of a magnetic field the interference pattern emerges from the relative phases of ψ_1 and ψ_{s2} in

$$\psi = \psi_1 + \psi_2 = R_1 e^{iS_1} + R_2 e^{iS_2} = e^{iS_1}\,(R_1 + e^{i(S_2 - S_1)}\,R_2) \tag{16A-13}$$

In the presence of a solenoid, (16A-13) is replaced by

$$\begin{aligned}\psi &= e^{-\frac{ie}{\hbar}\int_1 d\boldsymbol{\ell}'\cdot\mathbf{A}(\mathbf{r}')}\,\psi_1 + e^{-\frac{ie}{\hbar}\int_2 d\boldsymbol{\ell}'\cdot\mathbf{A}(\mathbf{r}')}\,\psi_2 \\ &= e^{-\frac{ie}{\hbar}\int_1 d\boldsymbol{\ell}'\cdot\mathbf{A}(\mathbf{r}')}\,e^{iS_1}(R_1 + R_2 e^{\,i(S_2 - S_1) + ie\Phi/\hbar})\end{aligned} \tag{16A-14}$$

If the solenoid is placed as in Fig. 16A-2(a), then no flux is enclosed by the paths of the electrons and there is no change in the interference pattern. If the flux is placed *between* the slits, as in Fig. 16A-2(b), then there is an additional contribution to the phase difference between ψ_1 and ψ_2, so that the optical path difference is changed by a constant that depends on the enclosed flux. This has the effect of shifting the peak interference pattern from the previous center, by an amount that depends on the enclosed flux. The first experimental confirmation of the effect is due to R. G. Chambers in 1960. The definitive experiments were done by A. Tonomura and collaborators in 1980.

The A-B paper generated a certain amount of controversy, because many people believed that since only electric and magnetic fields were "physical" nothing could depend on vector potentials. As was pointed out by M. Peshkin and others, the existence of the A-B effect is intimately tied to the quantization of angular momentum, and its *absence* would raise serious questions about quantum mechanics.

Consider a particle of charge $-e$ confined to a very thin torus with radius ρ lying in the x-y plane. A solenoid with radius $a \ll \rho$ is placed along the z-axis. The magnetic field B inside the solenoid points in the positive z-direction. The vector potential in a convenient, cylindrically symmetric gauge is

$$\mathbf{A} = -\frac{1}{2}\,\mathbf{r} \times \mathbf{B}$$

that is,

$$A_\rho = A_z = 0$$
$$A_\phi = \tfrac{1}{2}\rho B \qquad (16A\text{-}15)$$

The Hamiltonian operator has the form

$$\frac{1}{2m_e}(-i\hbar\,\nabla + e\mathbf{A})^2 + V(\rho)$$
$$= \frac{1}{2m_e}(-\hbar^2\nabla^2 - 2ie\hbar\mathbf{A}\cdot\nabla + e^2A^2) + V(\rho) \qquad (16A\text{-}16)$$
$$= -\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right) - \frac{ie\hbar B}{2m_e}\frac{\partial}{\partial\phi} + \frac{e^2B^2\rho^2}{8m_e} + V(\rho)$$

Now

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad (16A\text{-}17)$$

so that the Hamiltonian operator takes the form

$$-\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) + \frac{L_z^2}{2m_e\rho^2} + \frac{eBL_z}{2m_e} + \frac{e^2B^2\rho^2}{8m_e} + V(\rho)$$
$$= -\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) + \frac{1}{2m_e\rho^2}\left(L_z + \frac{eB\rho^2}{2}\right)^2 + V(\rho) \qquad (16A\text{-}18)$$

Let us now take$V(\rho)$ such that the electron is confined to a torus, so narrow that we may treat ρ as *constant*. In that case the Hamiltonian becomes

$$H = \frac{1}{2m_e\rho^2}\left(L_z + \frac{e\Phi}{2\pi}\right)^2 \qquad (16A\text{-}19)$$

aside from a constant. The eigenfunctions of H are eigenfunctions of L_z. With

$$L_z\psi = m\hbar\,\psi \qquad (16A\text{-}20)$$

where $m = 0, \pm 1, \pm 2, \ldots$, we find the energy eigenvalues to be

$$E = \frac{\hbar^2}{2m_e\rho^2}\left(m + \frac{e\Phi}{2\pi\hbar}\right)^2 \qquad (16A\text{-}21)$$

It is clear that the energy depends on the flux, even though the electron wave function nowhere overlaps the external magnetic field. To avoid this one would have to abandon quantization of angular momentum when the *x-y* plane has "holes" in it, or one would have to abandon the Schrödinger equation, or the measurability of energies (or rather energy differences) in the presence of such confined fluxes. There is no question of the correctness of this effect.

There is an interesting sidelight to the A-B effect. Consider a ring made of a superconductor placed in an external magnetic field, at a temperature above the critical temperature T_c below which the material becomes superconducting. When the temperature is lowered the superconductor *expels* the magnetic field (Fig. 16A-3) except for a thin surface layer so that $\mathbf{B} = 0$ inside the material (this is the so-called *Meissner effect*),[2] and

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[2]Chapter 21 in the *Feynman Lectures on Physics, Vol III* contains an excellent discussion of these macroscopic manifestations of quantum mechanics.

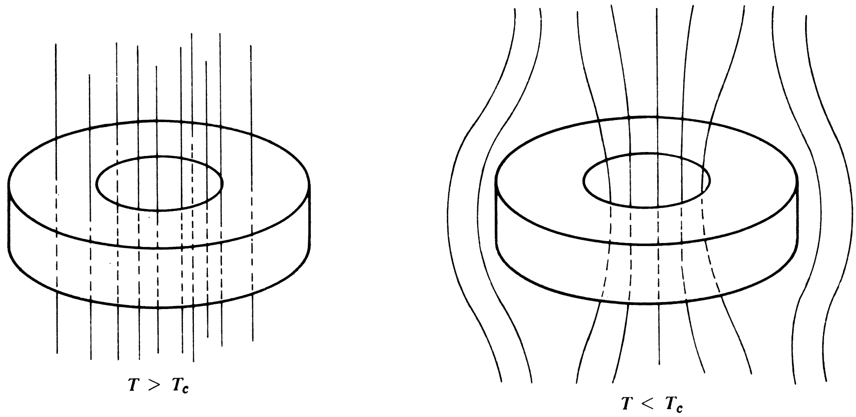

Figure 16A-3 A superconductor at temperature $T > T_c$ (the critical temperature) acts like any other metal, and magnetic flux lines can penetrate it. When the temperature is lowered until $T < T_c$, the ring becomes superconducting, and expels magnetic flux lines. Some of these become trapped inside the ring. It is the trapped flux that is found to be quantized.

magnetic flux is trapped inside the ring. The wave function of the superconductor is single-valued, and this implies that the phase factor that appears in (16A-10), when taken around a path that is inside the superconducting ring and encircles the flux region, must be unity. This implies that

$$e^{ie\Phi/\hbar} = 1 \tag{16A-22}$$

Thus the flux inside the ring is *quantized*, with

$$\frac{e\Phi}{\hbar} = 2\pi n \tag{16A-23}$$

This is almost right. The only modification that must be made is that the superconductor consists of a *condensate* of “correlated electron pairs,” so that the appropriate charge that appears in eq. (16A-23) is $2e$, and the flux quantization reads

$$\Phi = \frac{2\pi\hbar}{2e}\, n \tag{16A-24}$$

where n is an integer. The effect has been measured and the prediction (16A-24) was confirmed.

Supplement 16-B

A Little About Bessel Functions

The solution of the equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n^2}{z^2}\right)u = 0$$

with n integral, are known as Bessel functions, for the regular solutions

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{l=0}^{\infty} \frac{(iz/2)^{2l}}{l!(n+l)!}$$

and Neumann functions for the irregular solutions

$$N_n(z) = \frac{2}{\pi} J_n(z) \log\frac{\gamma z}{2} - \frac{1}{\pi}\left(\frac{z}{2}\right)^n \sum_{l=0}^{\infty} \frac{(iz/2)^{2l}}{l!(n+l)!} a_{nl} - \frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{l=0}^{n-1} \frac{(n-l-1)!}{l!}\left(\frac{z}{2}\right)^{2l}$$

$$(\log \gamma = 0.5772\ldots) \qquad a_{nl} = \left(\sum_{m=1}^{l} \frac{1}{m} + \sum_{m=1}^{l+n} \frac{1}{m}\right)$$

They have the asymptotic behavior

$$J_n(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\left[1 + 0\left(\frac{1}{z}\right)\right]$$

$$N_n(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\left[1 + 0\left(\frac{1}{z}\right)\right]$$

A detailed discussion of their properties may be found in any book on the special functions of mathematical physics.