Supplement 18-A

Quantizing the Electromagnetic Field Without Frills

Our goal is to express the electromagnetic field in terms of photons. In terms of the vector potential $\mathbf{A}(\mathbf{r}, t)$ we write

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$
$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \tag{18A-1}$$

As elsewhere in the book, we work in the gauge $\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$; we have also made the choice $\phi(\mathbf{r}) = 0$, possible in the absence of charges.

The energy carried by the electromagnetic field is

$$H = \frac{\varepsilon_0}{2} \int d^3r(\mathbf{E}^2(\mathbf{r}, t) + c^2\mathbf{B}^2(\mathbf{r}, t)) \tag{18A-2}$$

Let us now expand $\mathbf{A}(\mathbf{r}, t)$ in a Fourier series in a cubical box of volume $V = L^3$. We write

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},\lambda} f(k)\varepsilon_\lambda\,(\mathbf{k})\left\{A_\lambda^+(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{A}_\lambda(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\} \tag{18A-3}$$

This will obey the wave equation provided that

$$\omega^2 = \mathbf{k}^2c^2 \tag{18A-4}$$

The gauge-fixing condition implies that

$$\mathbf{k} \cdot \varepsilon^{(\lambda)}(\mathbf{k}) = 0 \tag{18A-5}$$

This means that the *polarization vectors* $\varepsilon^{(\lambda)}(\mathbf{k})$ ($\lambda = 1, 2,$) are perpendicular to the direction of propagation of the wave $\mathbf{k}$; that is, the polarization is *transverse*.

Let us now calculate the energy in terms of the $\mathbf{A}_\lambda(\mathbf{k})$ and $A_\lambda^+(\mathbf{k})$. We get for the first term involving the electric field,

$$\frac{1}{V}\int d^3r\sum_{\mathbf{k},\lambda}\sum_{\mathbf{q},\lambda'} f(\mathbf{k})\,f(\mathbf{q})\;\varepsilon_\lambda\,(\mathbf{k})\;\varepsilon_{\lambda'}\,(\mathbf{q})\left\{-i\omega A_\lambda^+\,(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + i\omega A_\lambda\,(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\}$$
$$\times\left\{-i\omega'\,A_{\lambda'}^+(\mathbf{q})e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega' t)} + i\omega'\,A_{\lambda'}(\mathbf{q})e^{i(\mathbf{q}\cdot\mathbf{r}-\omega' t)}\right\} \tag{18A-6}$$

We use the notation $\omega' = |\mathbf{q}|c$. The values of $\mathbf{k}$ and $\mathbf{q}$ are determined by the fact that we are expanding in a box, and we choose periodic boundary conditions. Thus in any direction we require that

$$e^{ikx} = e^{ik(x+L)} \tag{18A-7}$$

so that $k_1L = 2\pi n_1$, and so on. Thus the summations are over integers n_1, n_2, $n_3 = 1, 2, 3, \ldots$. We do not need to sum over negative integers, since these are contained in the $\mathbf{A}^+$ terms. Now

$$\frac{1}{V}\int d^3 r e^{i(\mathbf{k}+\mathbf{q})\mathbf{r}} = 0$$
$$\frac{1}{V}\int d^3 r e^{i(\mathbf{k}-\mathbf{q})\mathbf{r}} = \delta_{kq} \tag{18A-8}$$

We also choose

$$\varepsilon_\lambda(\mathbf{k}) \bullet \varepsilon_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'} \tag{18A-9}$$

so that the two polarization directions corresponding to $\lambda = 1, 2$ are perpendicular to each other. This yields the result

$$H_{(E)} = \frac{\varepsilon_0}{2}\sum_{\mathbf{k},\lambda} f^2(\mathbf{k})\,\omega^2(A_\lambda(\mathbf{k})A_\lambda^+(\mathbf{k}) + A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k})) \tag{18A-10}$$

The second term, involving **B**, requires the calculation of

$$\frac{1}{V}\sum_{\mathbf{k},\lambda}\sum_{\mathbf{q},\lambda'} f(\mathbf{k})\,f(\mathbf{q})(\mathbf{k}\times\varepsilon_\lambda(\mathbf{k}))\bullet(\mathbf{q}\times\varepsilon_{\lambda'}(\mathbf{q})$$
$$\times\left\{iA_\lambda^+(\mathbf{k})\,e^{-i(\mathbf{k}\bullet\mathbf{r}-\omega t)} - iA_\lambda(\mathbf{k})\,e^{i(\mathbf{k}\bullet\mathbf{r}-\omega t)}\right\}\left\{iA_{\lambda'}^+(\mathbf{q})e^{-i(\mathbf{q}\bullet\mathbf{r}-\omega' t)} - iA_{\lambda'}(\mathbf{q})\,e^{i(\mathbf{q}\bullet\mathbf{r}-\omega' t)}\right\} \tag{18A-11}$$

The integration over all the spatial coordinates again yields

$$\frac{1}{V}\int d^3 r e^{i(\mathbf{k}+\mathbf{q})\bullet\mathbf{r}} = 0$$
$$\frac{1}{V}\int d^3 r e^{i(\mathbf{k}-\mathbf{q})\bullet\mathbf{r}} = \delta_{\mathbf{kq}} \tag{18A-12}$$

The vector identity

$$(\mathbf{k}\times\varepsilon_\lambda(\mathbf{k}))\bullet(\mathbf{k}\times\varepsilon_{\lambda'}(\mathbf{k})) = k^2\varepsilon_\lambda(\mathbf{k})\bullet\varepsilon_{\lambda'}(\mathbf{k}) - (\mathbf{k}\bullet\varepsilon_\lambda(\mathbf{k}))(\mathbf{k}\bullet\varepsilon_{\eta'}(\mathbf{k})) \tag{18A-12}$$

with the help of the tranversality condition and $\varepsilon_\lambda(\mathbf{k}) \bullet \varepsilon_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'}$ yields $k^2\delta_{\lambda\lambda'}$. This means that the part of H involving B, when multiplied by c^2, yields the same factor as the $\mathbf{E}^2$ term. We thus get

$$H = \varepsilon_0\sum_{\mathbf{k},\lambda} f^2(\mathbf{k})\,\omega^2\left\{A_\lambda(\mathbf{k})\,A_\lambda^+(\mathbf{k}) + A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k})\right\} \tag{18A-13}$$

The form looks very much like the sum of terms in the simple harmonic oscillator. In fact, had we chosen $f(\mathbf{k})$ such that

$$\varepsilon_0 f^2(\mathbf{k})\,\omega^2 = \frac{1}{2}\hbar\omega$$

that is,

$$f(\mathbf{k}) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega}}$$

we would have obtained

$$H = \sum_{\mathbf{k},\lambda}\frac{1}{2}\hbar\omega\left\{A_\lambda(\mathbf{k})\,A_\lambda^+(\mathbf{k}) + A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k})\right\} \tag{18A-14}$$

Let us now assume that the $A_\lambda(\mathbf{k})$ and $A_\lambda^+(\mathbf{k})$ are *operators* that obey the same commutation relations as the operators A and A^+ for the simple harmonic oscillator problem; that is

$$[A_\lambda(\mathbf{k}), A_{\lambda'}^+(\mathbf{q})] = \delta_{\lambda\lambda'}\delta_{\mathbf{kq}} \tag{18A-15}$$

Then we get

$$H = \sum_{\mathbf{k},\lambda} \hbar\omega\,((A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k}) + \tfrac{1}{2}) \tag{18A-16}$$

Actually the second term $\Sigma_{\mathbf{k},\lambda}\hbar\omega/2$ is infinite. We sweep this problem under the rug by observing that all energy measurements are measurements of energy differences. We thus concentrate on

$$H = \sum_{\mathbf{k},\lambda} \hbar\omega A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k}) \tag{18A-17}$$

We may now go through the same steps that we did with the harmonic oscillator. For each value of $\mathbf{k}$ and λ we have creation and annihilation operators, and for each value of $\mathbf{k}$ and λ we have states of zero, one, two, . . . photons. The zero photon state, the *vacuum state*, is $|0\rangle$, defined by

$$A_\lambda(\mathbf{k})|0\rangle = 0 \tag{18A-18}$$

A state with n photons of momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$ is given by

$$\frac{1}{\sqrt{n!}}(A_\lambda^+(\mathbf{k}))^n|0\rangle \tag{18A-19}$$

What we have done is to decompose the electromagnetic field into modes, each of which represents photons. Thus

$$\mathbf{E}(\mathbf{r}, t) = \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t} = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},\lambda}\sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\,\varepsilon_\lambda(\mathbf{k})\left\{-iA_\lambda^+(\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + iA_\lambda(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\} \tag{18A-20}$$

will annihilate or create a single photon.

As a check we can calculate the momentum carried by the electromagnetic field. We need to calculate

$$\mathbf{P} = \varepsilon_0\int d^3r(\mathbf{E}(\mathbf{r}, t)\times\mathbf{B}(\mathbf{r}, t)) \tag{18A-21}$$

Using the expressions obtained above, we find, after a page of algebra, that

$$\mathbf{P} = \sum_{\mathbf{k},\lambda}\hbar\mathbf{k}(A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k}) + 1) = \sum_{\mathbf{k},\lambda}\hbar\mathbf{k}A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k}) \tag{18A-22}$$

The last step follows from the fact that $\Sigma_{\mathbf{k}}\,\mathbf{k} = 0$ by symmetry. We may thus interpret the product $A_\lambda^+(\mathbf{k})\,A_\lambda(\mathbf{k}) \equiv N_\lambda(\mathbf{k})$ as the operator representing the number of photons of momentum k and polarization λ.

Supplement 18-B

Details of the Three-Level System

The three-level system has many interesting features, so that we discuss it in some detail The system of three levels, A, B, C with energies $\hbar\omega_a > \hbar\omega_b > \hbar\omega_c$ is placed in a set of electric fields. One of them is characterized by a frequency ω_1 close to the difference $\omega_a - \omega_b$, and the other by a frequency ω_2, close to the difference $\omega_a - \omega_c$. The perturbing Hamiltonian is

$$H_1 = eE_1 x \cos \omega_1 t + eE_2 x \cos \omega_2 t \tag{18B-1}$$

In matrix form, the only nonvanishing elements are taken to be $\langle a|x|b\rangle$ and $\langle a|x|c\rangle$, which can be taken to be real. We introduce the notation

$$\begin{aligned} W_{1ab} &= eE_1\langle a|x|b\rangle \\ W_{2ab} &= eE_2\langle a|x|b\rangle \\ W_{1ac} &= eE_1\langle a|x|c\rangle \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{18B-2}$$

and so on. With this notation the matrix representation of H_1 has the form

$$\begin{pmatrix} 0 & W_{1ab}\cos\omega_1 t + W_{2ab}\cos\omega_2 t & W_{1ac}\cos\omega_1 t + W_{2ac}\cos\omega_2 t \\ W_{1ab}\cos\omega_1 t + W_{2ab}\cos\omega_2 t & 0 & 0 \\ W_{1ac}\cos\omega_1 t + W_{2ac}\cos\omega_2 t & 0 & 0 \end{pmatrix} \tag{18B-3}$$

We next need to calculate $e^{iH_0t/\hbar}H_1e^{-iH_0t/\hbar}$. To get this we pre-multiply the matrix for H_1

$$e^{iH_0t/\hbar} = \begin{pmatrix} e^{i\omega_a t} & 0 & 0 \\ 0 & e^{i\omega_b t} & 0 \\ 0 & 0 & e^{i\omega_c t} \end{pmatrix} \tag{18B-4}$$

and post-multiply it by the hermitian conjugate matrix. Some algebra yields the matrix

$$V(t) = \begin{pmatrix} 0 & X & Y \\ X^* & 0 & 0 \\ Y^* & 0 & 0 \end{pmatrix} \tag{18B-5}$$

with

$$\begin{aligned} X &= (W_{1ab}\cos\omega_1 t + W_{2ab}\cos\omega_2 t)\, e^{i(\omega_a-\omega_b)t} \\ Y &= (W_{1ac}\cos\omega_1 t + W_{2ac}\cos\omega_2 t)\, e^{i(\omega_a-\omega_c)t} \end{aligned} \tag{18B-6}$$

We now again apply the rotating wave approximation, with $\omega_a - \omega_b - \omega_1 = \delta_1$, and $\omega_a - \omega_c - \omega_2 = \delta_2$ being the only terms that we keep in this approximation. This means that in X and Y we decompose the cosines, and only keep the terms below:

$$X = \frac{1}{2} W_{1ab} e^{i\delta_1 t}$$
$$Y = \frac{1}{2} W_{2ac} e^{i\delta_2 t} \tag{18B-7}$$

Let us take the state vector as represented by the column vector

$$|\psi_1(t)\rangle \rightarrow \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} \tag{18B-8}$$

The set of equations to be solved is

$$\begin{aligned} i\hbar \frac{da(t)}{dt} &= \frac{1}{2} W_{1ab} e^{i\delta_1 t}\, b(t) + \frac{1}{2} W_{2ac} e^{i\delta_2 t}\, c(t) \\ i\hbar \frac{db(t)}{dt} &= \frac{1}{2} W_{1ab} e^{-i\delta_1 t}\, a(t) \\ i\hbar \frac{dc(t)}{dt} &= \frac{1}{2} W_{2ac} e^{-i\delta_2 t}\, a(t) \end{aligned} \tag{18B-9}$$

Let us write

$$\begin{aligned} B(t) &= e^{i\delta_1 t} b(t) \\ C(t) &= e^{i\delta_2 t} c(t) \end{aligned} \tag{18B-10}$$

In terms of these, the equations become

$$\begin{aligned} i\hbar \frac{da(t)}{dt} &= \frac{W_{1ab}}{2} B(t) + \frac{W_{2ac}}{2} C(t) \\ i\hbar \frac{dB(t)}{dt} + \hbar\delta_1 B(t) &= \frac{W_{1ab}}{2} A(t) \\ i\hbar \frac{dC(t)}{dt} + \hbar\delta_2 C(t) &= \frac{W_{2ac}}{2} A(t) \end{aligned} \tag{18B-11}$$

Let us now assume the time dependence $e^{i\Omega t}$ in all the terms. We then get from the various equations

$$\begin{aligned} -\hbar\Omega a(0) &= \frac{W_{1ab}}{2} B(0) + \frac{W_{2ac}}{2} C(0) \\ -\hbar(\Omega - \delta_1)\, B(0) &= \frac{W_{1ab}}{2} a(0) \\ -\hbar(\Omega - \delta_2)\, C(0) &= \frac{W_{2ac}}{2} a(0) \end{aligned} \tag{18B-12}$$

This leads to a cubic equation for Ω, as might have been expected. The equation reduces to

$$\Omega(\Omega - \delta_1)(\Omega - \delta_2) = \left(\frac{W_{1ab}}{2\hbar}\right)^2 (\Omega - \delta_2) + \left(\frac{W_{2ac}}{2\hbar}\right)^2 (\Omega - \delta_1) \tag{18B-13}$$

We can greatly simplify matters by assuming *perfect tuning* so that $\delta_1 = \delta_2 = 0$. In that case the equation has simple roots: $\Omega = 0$, $\Omega = r$, $\Omega = -r$, where

$$r = \sqrt{\left(\frac{W_{1ab}}{2\hbar}\right)^2 + \left(\frac{W_{2ac}}{2\hbar}\right)^2} \tag{18B-14}$$

Thus we have

$$\begin{aligned} a(t) &= a_0 + a_+ e^{irt} + a_- e^{-irt} \\ b(t) &= b_0 + b_+ e^{irt} + b_- e^{-irt} \\ c(t) &= c_0 + c_+ e^{irt} + c_- e^{-irt} \end{aligned} \tag{18B-15}$$

The nine parameters will be determined by the equations (18B-12).

We now write the solutions in terms of the eigenstates corresponding to the different eigenvalues

$$\begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} + \begin{pmatrix} a_+ \\ b_+ \\ c_+ \end{pmatrix} e^{irt} + \begin{pmatrix} a_- \\ b_- \\ c_- \end{pmatrix} e^{-irt} \tag{18B-16}$$

To find the normalized eigenstates we proceed as follows:

$\Omega = 0$: We have $a_0 = 0$, and $W_{1ab}\, b_0 + W_{2ac}\, c_0 = 0$. The normalized solutions is therefore

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ W_{2ac}/2\hbar r \\ -W_{1ab}/2\hbar r \end{pmatrix} \tag{18B-17}$$

This is normalized to unity since it $W_{2ac}^2 + W_{1ab}^2 = 4\hbar^2 r^2$.

For $\Omega = r$, we must satisfy

$$\begin{aligned} -\hbar r a_+ &= \frac{W_{1ab}}{2} b_+ + \frac{W_{2ac}}{2} c_+ \\ -\hbar r b_+ &= \frac{W_{1ab}}{2} a_+ \\ -\hbar r c_+ &= \frac{W_{2ac}}{2} a_+ \end{aligned} \tag{18B-18}$$

The first equation is automatically satisfied if the other two are, which is to be expected, since a_+ is to be determined by normalization. A little algebra shows that

$$\begin{pmatrix} a_+ \\ b_+ \\ c_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -W_{1ab}/2\hbar r \\ -W_{2ac}/2\hbar r \end{pmatrix} \tag{18B-19}$$

The case for $\Omega = -r$ is easily solved by just changing the sign of r. We get

$$\begin{pmatrix} a_- \\ b_- \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ W_{1ab}/2\hbar r \\ W_{2ac}/2\hbar r \end{pmatrix} \tag{18B-20}$$

It is easy to check that the three eigenvectors are mutually orthogonal.

The general solution is

$$\begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} = \alpha_0 \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} + \alpha_+ \begin{pmatrix} a_+ \\ b_+ \\ c_+ \end{pmatrix} e^{irt} + \alpha_- \begin{pmatrix} a_- \\ b_- \\ c_- \end{pmatrix} e^{-irt} \tag{18B-21}$$

and the coefficients (α_0, $\alpha_+\alpha_-$) are determined by the initial conditions. At time $t = 0$ we have

$$\begin{aligned} a(0) &= \frac{1}{\sqrt{2}}\alpha_+ + \frac{1}{\sqrt{2}}\alpha_- \\ b(0) &= \frac{W_{2ac}}{2\hbar r}\alpha_0 - \frac{W_{1ab}}{2\sqrt{2}\hbar r}\alpha_+ + \frac{W_{1ab}}{2\sqrt{2}\hbar r}\alpha_- \\ c(0) &= \frac{W_{1ab}}{2\hbar r}\alpha_0 - \frac{W_{2ac}}{2\sqrt{2}\hbar r}\alpha_+ + \frac{W_{2ac}}{2\sqrt{2}\hbar r}\alpha_- \end{aligned} \tag{18B-22}$$

Dark States

Consider a configuration in which the $|b\rangle$ and the $|c\rangle$ states lie close together (Fig. 18-3*a*) and

$$W_{1ab} = W_{2ac} \tag{18B-23}$$

We now take our initial state to be

$$|\psi_1(0)\rangle = \frac{1}{\sqrt{2}}(|b\rangle - |c\rangle) \tag{18B-24}$$

This implies that $a(0) = 0$; $b(0) = -c(0) = 1/\sqrt{2}$. We now solve for ($\alpha_0$, $\alpha_+\alpha_-$), and get

$$\begin{aligned} \alpha_0 &= \frac{W_{1ab} + W_{2ac}}{2\sqrt{2}\hbar r} = \frac{W_{1ab}}{\sqrt{2}\hbar r} \\ \alpha_+ &= -\alpha_- = \frac{W_{2ac} - W_{1ab}}{4\hbar r} = 0 \end{aligned} \tag{18B-25}$$

This implies that $a(t) = 0$ for all times. Thus the state $|a\rangle$ is *never* excited, and is therefore called a *dark state*. The reason it is inaccessible is that the amplitudes for exciting from the $|b\rangle$ and $|c\rangle$ states interfere destructively.

Electromagnetically Induced Transparency

Consider, next, a situation in which the $|a\rangle$ and $|b\rangle$ states are strongly coupled by an electromagnetic field, while $|a\rangle$ and $|c\rangle$ are weakly coupled. What this implies is that

$$|W_{1ab}| \gg |W_{2ac}| \tag{18B-26}$$

One can show that under these circumstances the state $|a\rangle$ is very unlikely to be excited, and this means that photons cannot be absorbed by a $|c\rangle \rightarrow |a\rangle$ transition.

We take for our initial condition $a(0) = b(0) = 0$. This implies that

$$\begin{aligned} \alpha_- &= -\alpha_+ \\ \alpha_+ &= \frac{W_{2ac}}{\sqrt{2}\,W_{1ab}}\alpha_0 \end{aligned} \tag{18B-27}$$

Furthermore, since $a(0) = b(0) = 0$, we must necessarily have $|c(0)| = 1$. Since

$$c(0) = \frac{2\hbar r}{W_{1ab}}\alpha_0$$

we may choose for convenience

$$\alpha_0 = \frac{W_{1ab}}{2\hbar r} \tag{18B-28}$$

Now, at a later time t, we get

$$a(t) = \frac{\alpha_+}{\sqrt{2}}(e^{irt} - e^{-irt}) = i\,\frac{W_{2ac}}{2\hbar r}\sin rt \tag{18B-29}$$

Thus the probability of finding the system in the state $|a\rangle$ at a later time is

$$P_a(t) = |a(t)|^2 = \frac{W_{2ac}^2}{W_{2ac}^2 + W_{1ab}^2}\,\sin^2 rt \tag{18B-30}$$

Because of the condition (18B-26) the probability of exciting the state $|a\rangle$ is very small. This, however, implies that photons cannot be absorbed through the mechanism of exciting state $|a\rangle$, so the material becomes *transparent* at the frequency corresponding to this energy difference.